Cocompleteness in simplicial homotopy type theory

Daniel Gratzer[1] Jonathan Weinberger[2] **Ulrik Buchholtz**[3]

[1]Aarhus University

[2]Chapman University

[3]University of Nottingham

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Motivation for simplicial type theory (STT)

- Last year at TYPES: Motivation for simplicial type theory via circumventing the coherence problem: Formalize theory of $(\infty, 1)$-categories in HoTT/UF.
- This year, directed type theory is in the air!
- But directed type theory is also hard.
- Simplicial type theory offers you a bargain:

You can do directed type theory *today*
in ordinary[1] type theory,
for the small price of proving a few simple propositions!

[1] T&Cs apply, some modalities may creep in

Basics of simplicial type theory

Riehl–Shulman [RS17] introduced simplicial type theory. The idea is to interpret (homotopy) type theory in simplicial objects $\mathrm{s}\mathcal{E}$ of a model of (homotopy) type theory $\mathcal{E}$. Here we have the simplices Δ^n, generated from an interval type $\mathbb{I}$ (totally ordered with $0 \neq 1$).

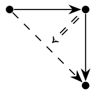
Using $\mathbb{I}$, we get the type of arrows in any type, $X^{\mathbb{I}}$, and we get functoriality for free by composition: $\mathbb{I} \to X \to Y$.

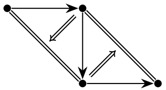
Definition X is *Segal* if $X^{\Delta^2} \to X^{\Lambda^2_1}$ is an equivalence.

Definition A Segal type X is *Rezk* if $X^{\mathbb{E}} \to X$ is an equivalence, for $\mathbb{E}$ the "walking equivalence".

Definition Functions $f : C \to D$ and $g : D \to C$ are adjoint when equipped with $\iota : \prod_{c,d} \hom(f(c), d) \simeq \hom(c, g(d))$.

$\Lambda^2_1 \hookrightarrow \Delta^2$





$\mathbb{E}$

Further developments in simplicial type theory

- Fibered category theory [BW23]
- (co)limits and exponentiable functors, Bardomiano Martínez [Bar22; Bar24]
- Prototype proof assistant Rzk with formalization of the fibrational Yoneda lemma [Kud23; KRW04]
- Directed univalent universe of spaces via LOPS construction [Lic+18]; bicubical model by Licata–Weaver [WL20; Wea24], triangulated type theory [GWB24], with duality axiom [Ble23; Wil25] and cohesion-style axioms [MR23] – directed structure identity principle, TYPES 2024
- Yoneda lemma via twisted arrow modality, Kan extensions, and Quillen's Theorem A [GWB25] – LICS 2025
- Today: Decomposition of cocompleteness, applications to synthetic stable homotopy theory

Sifted colimits

Definition A crisp category C is sifted if $\varinjlim_C : \mathcal{S}^C \to \mathcal{S}$ preserves finite products.

Lemma A crisp category C is sifted if and only if for all $n : \mathrm{Nat}$ the map $C \to C^n$ is right cofinal[2].

The proof goes via Quillen's Theorem A: A functor $f :_\flat C \to D$ is right cofinal iff $L_{\mathbb{I}}(C_{d/})$ is contractible iff $\varinjlim_D X \to \varinjlim_D X \circ f$ is an equivalence for all $X :_\flat D \to \mathcal{S}$.

Theorem If C has finite coproducts and sifted colimits, then it is cocomplete.

Proposition $\langle \mathfrak{o} \mid \Delta \rangle$ is sifted.

[2] Ie, left orthogonal to right fibrations; aka final, cofinal, and left cofinal – it's a mess.

Filtered colimits

Definition The finite categories are those generated by pushouts starting from 0, 1, and $\mathbb{I}$.

Definition A crisp category C is *filtered* if $\varinjlim : \mathcal{S}^C \to \mathcal{S}$ preserves finite limits.

It suffices to preserve pullbacks and the terminal object.

Definition A crisp category C is *weakly filtered* if $C \to C^X$ is right cofinal for all finite X.

It's easy to see that filtered categories are weakly filtered. The converse is due to Sattler and Wärn [SW25], whose foundation-agnostic argument directly translates to STT.

Theorem If C has finite and filtered colimits, then it is cocomplete.

Stable homotopy theory

Stable homotopy theory concerns phenomena in the limit of suspending sufficiently often, turning types into symmetric ∞-groups (recall Freudenthal suspension theorem).

Definition The *category of spectra* is $\mathrm{Sp} = \varprojlim(\mathcal{S}_\star \leftarrow \mathcal{S}_\star \leftarrow \dots)$.

Lemma Sp is closed under finite limits and filtered colimits.

Following an argument of Cnossen [Cno25] we get:

Lemma $\Omega : \mathrm{Sp} \to \mathrm{Sp}$ is an equivalence.

It's clear that shifting down is an equivalence, but it's not immediate that Ω agrees with this. But that's easier when looking at the even numbers (right cofinal in Nat)!

Lemma Sp is finitely cocomplete and pushout squares and pullback squares coincide.

Corollary Sp is cocomplete.

Smash product of spectra

Ordinary homology $\mathrm{H}(-; R)$ is now definable as $(1 \mapsto \mathrm{H}R)_!$ – the unique cocontinuous functor $\mathcal{S} \to \mathsf{Sp}$ that sends 1 to $\mathrm{H}R$.

Theorem The family of functors $H_i : \mathcal{S} \to \mathrm{Ab}$ defined by $H_i(X) = \pi_i(\mathrm{H}(X; R))$ satisfies the Eilenberg-Steenrod axioms.

Directed univalence allows us to define the smash product as a functor $- \wedge - : \mathcal{S}_* \times \mathcal{S}_* \to \mathcal{S}_*$ and transfer results from the HoTT definition, like associativity Ljungström [Lju24].

Definition If $X, Y : \mathsf{Sp}$, we define $X \otimes Y = \varinjlim_{i,j:\mathsf{Nat}} \Omega^{i+j}\Sigma^{\infty}(X_i \wedge Y_j)$.

This uses directed univalence to establish functoriality.

Some higher algebra

We can begin to play with some higher algebra using the idea of *animation*, a term due to Dustin Clausen [ČS24]:

For a cocomplete 1-category C generated under colimits by its compact projectives C^{sfp}, e.g.,

- Set generated by finite sets,
- Group generated by free groups on finite generating sets,
- Ab generated by f.g. free abelian groups,
- ...

define its *animation* $\mathrm{Ani}(C)$ to be the free sifted colimit completion of C^{sfp}.

Example $\mathrm{Ani}(\mathrm{Ab})$ is a possible definition of the category of connective $\mathrm{H}\mathbb{Z}$-modules.

Next steps for simplicial type theory

- Straightening–unstraightening for the universe of categories (the $(\infty, 1)$-category of $(\infty, 1)$-categories).
  Unlocks many further constructions:
  - Monads, operads, (symmetric) monoidal categories, (∞, n)-categories, ...
- $(\mathrm{Sp}, \otimes, \mathbb{S})$ as a symmetric monoidal category (or better, the monoidal unit in the symmetric monoidal category of presentable stable categories)
- Higher topos theory
- Metatheory of type theory *internally in STT*
- Interpretations of ($(\infty, 1)$-localic) directed type theories
- Computational version, metatheory of *of STT*
- More formalization
- ...

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