Girard – Reynolds :

System F

in

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The polymorphic λ-calculus, λ2

Types ::= $\mathfrak{X}$ | $\sigma \to \tau$ | $\forall \alpha, \sigma$

- $\mathfrak{X}$ — type variables
- $\sigma \to \tau$ — arrow types
- $\forall \alpha, \sigma$ — abstraction (polymorphic) types

Term rules:

$$\dfrac{\begin{matrix} & x : \tau \\ \vdots & \vdots \\ N : \tau \to \sigma & M : \tau \end{matrix}}{NM : \sigma} \to E \qquad \dfrac{\begin{matrix} [x : \tau] \\ \vdots \\ M : \sigma \end{matrix}}{\lambda x{:}\tau . M : \tau \to \sigma} \to I$$

$$\dfrac{\begin{matrix} \vdots \\ M : \forall \alpha . \sigma \end{matrix}}{M\tau : \sigma[\tau/\alpha]} \forall E \qquad \dfrac{\begin{matrix} \Gamma \\ \vdots \\ M : \sigma \end{matrix}}{\Lambda \alpha . M : \forall \alpha . \sigma} \forall I$$

$(\alpha \notin FV(\Gamma))$

note: terms can <u>de</u>pe<u>nd</u> on types!

Example derivation

$$\dfrac{[f : \forall\alpha.\alpha\to\beta\to\alpha]}{f\,\alpha : \alpha\to\beta\to\alpha}\ \forall E$$

$$\dfrac{f\,\alpha : \alpha\to\beta\to\alpha \qquad [x:\alpha]}{f\,\alpha\,x : \beta\to\alpha}\ \to E$$

$$\dfrac{[f : \forall\alpha.\alpha\to\beta\to\alpha]}{f\,(\beta\to\alpha) : (\beta\to\alpha)\to\beta\to\beta\to\alpha}\ \forall E$$

$$\dfrac{f\,(\beta\to\alpha) : (\beta\to\alpha)\to\beta\to\beta\to\alpha \qquad f\,\alpha\,x : \beta\to\alpha}{f\,(\beta\to\alpha)\,(f\,\alpha\,x) : \beta\to\beta\to\alpha}\ \to E$$

$$\dfrac{f\,(\beta\to\alpha)\,(f\,\alpha\,x) : \beta\to\beta\to\alpha}{\lambda x{:}\alpha.\, f\,(\beta\to\alpha)\,(f\,\alpha\,x) : \alpha\to\beta\to\beta\to\alpha}\ \to I, x$$

$$\dfrac{\lambda x{:}\alpha.\, f\,(\beta\to\alpha)\,(f\,\alpha\,x) : \alpha\to\beta\to\beta\to\alpha}{\Lambda\alpha.\lambda x{:}\alpha.\, f\,(\beta\to\alpha)\,(f\,\alpha\,x) : \forall\alpha.\alpha\to\beta\to\beta\to\alpha}\ \forall I$$

$$\dfrac{\Lambda\alpha.\lambda x{:}\alpha.\, f\,(\beta\to\alpha)\,(f\,\alpha\,x) : \forall\alpha.\alpha\to\beta\to\beta\to\alpha}{\lambda f{:}\forall\alpha.\alpha\to\beta\to\alpha.\ \Lambda\alpha.\lambda x{:}\alpha.\, f\,(\beta\to\alpha)\,(f\,\alpha\,x) : (\forall\alpha.\alpha\to\beta\to\alpha)\to\forall\alpha.\alpha\to\beta\to\beta\to\alpha}\ \to I, f$$

Expressive power

Absurdity: $\bot \triangleq \forall\alpha.\alpha$

$$\dfrac{M : \bot}{M\tau : \tau}\ \bot_i \quad (\forall E)$$

Conjunction: $\tau \wedge \sigma \triangleq \forall\alpha.(\tau\to\sigma\to\alpha)\to\alpha$

$$\left.\dfrac{\dfrac{\dfrac{\dfrac{[z : \tau\to\sigma\to\alpha] \quad M : \tau}{zM : \sigma\to\alpha} \qquad N : \sigma}{zMN : \alpha}}{\lambda z.\, zMN : (\tau\to\sigma\to\alpha)\to\alpha}}{\Lambda\alpha.\lambda z.\, zMN : \forall\alpha.(\tau\to\sigma\to\alpha)\to\alpha}\right\}\ \wedge I$$

$$\forall\alpha.(\tau\to\sigma\to\alpha)\to\alpha = \tau\wedge\sigma$$

$$\left.\dfrac{\dfrac{P : \tau\wedge\sigma}{P\tau : (\tau\to\sigma\to\tau)\to\tau} \qquad \dfrac{\dfrac{[x : \tau]}{\lambda y{:}\sigma.x : \sigma\to\tau}}{\lambda x{:}\tau.\lambda y{:}\sigma.x : \tau\to\sigma\to\tau}}{P\tau\ \lambda x{:}\tau.\lambda y{:}\sigma.x : \tau}\right\}\ \wedge E_L$$

Disjunction:

$$\tau \vee \sigma \triangleq \forall \alpha. (\tau \to \alpha) \to (\sigma \to \alpha) \to \alpha$$

Natural numbers:

$$\mathbb{N} \triangleq \forall \alpha. (\alpha \to \alpha) \to \alpha \to \alpha$$

$$3 \triangleq \Lambda \alpha. \lambda f : \alpha \to \alpha. \lambda x : \alpha. f (f (f x))$$

Lists:

$$\text{List}\ \tau \triangleq \forall \alpha. \alpha \to (\tau \to \alpha \to \alpha) \to \alpha$$

Existential:

$$\exists \alpha. \sigma \triangleq \forall \beta. (\forall \alpha. \sigma \to \beta) \to \beta$$

⋮

Inductive types

Second-order intuitionistic propositional logic

Usual rules plus:

$$\dfrac{\begin{matrix}\Gamma \\ \vdots \\ \varphi\end{matrix}}{\forall p.\,\varphi}\ \forall I \quad (p \notin FV(\Gamma))$$

$$\dfrac{\begin{matrix}\Gamma \\ \vdots \\ \forall p.\,\varphi\end{matrix}}{\varphi[\theta/p]}\ \forall E$$

$$\dfrac{\begin{matrix}\Gamma \\ \vdots \\ \varphi[\theta/p]\end{matrix}}{\exists p.\,\varphi}\ \exists I$$

$$\dfrac{\begin{matrix}\Gamma & & [\varphi] \\ \vdots & \vdots & \vdots \\ \exists p.\,\varphi & & \psi\end{matrix}}{\psi}\ \exists E \quad (p \notin FV(\Gamma, \psi))$$

Semantics:

Soundness & completeness for

- complete 2nd-order Kripke models
- all valuations in all complete Heyting algebras

Thm It is undecidable whether a given formula has a proof.

Properties of $\lambda 2$

Thm (Curry-Howard isomorphism)

$$\Gamma \vdash_{\lambda 2} M : \tau \iff |\Gamma| \vdash \tau$$

$\uparrow$ $(\forall, \to)$-fragment of 2nd int. prop. logic

Thm The type inhabitation problem for $\lambda 2$ is undecidable ($\forall \tau \exists M : \tau$ closed term)

Thm β-reduction for $\lambda 2$ is strongly normalizing.

Thm (Girard)
The class of functions $\mathbb{N} \to \mathbb{N}$ definable in $\lambda 2$ coincides with the class of provably recursive functions of second-order Peano Arithmetic.

Church vs. Curry-style

erasure map $|-| : \lambda 2 \to$ pure λ-calc: λ

$$|x| = x$$

$$|MN| = |M||N|$$

$$|\lambda x{:}\sigma.M| = \lambda x.|M|$$

$$|\Lambda\alpha.M| = |M|$$

$$|M\tau| = |M|$$

Prop: $M \in$ pure λ-calculus:

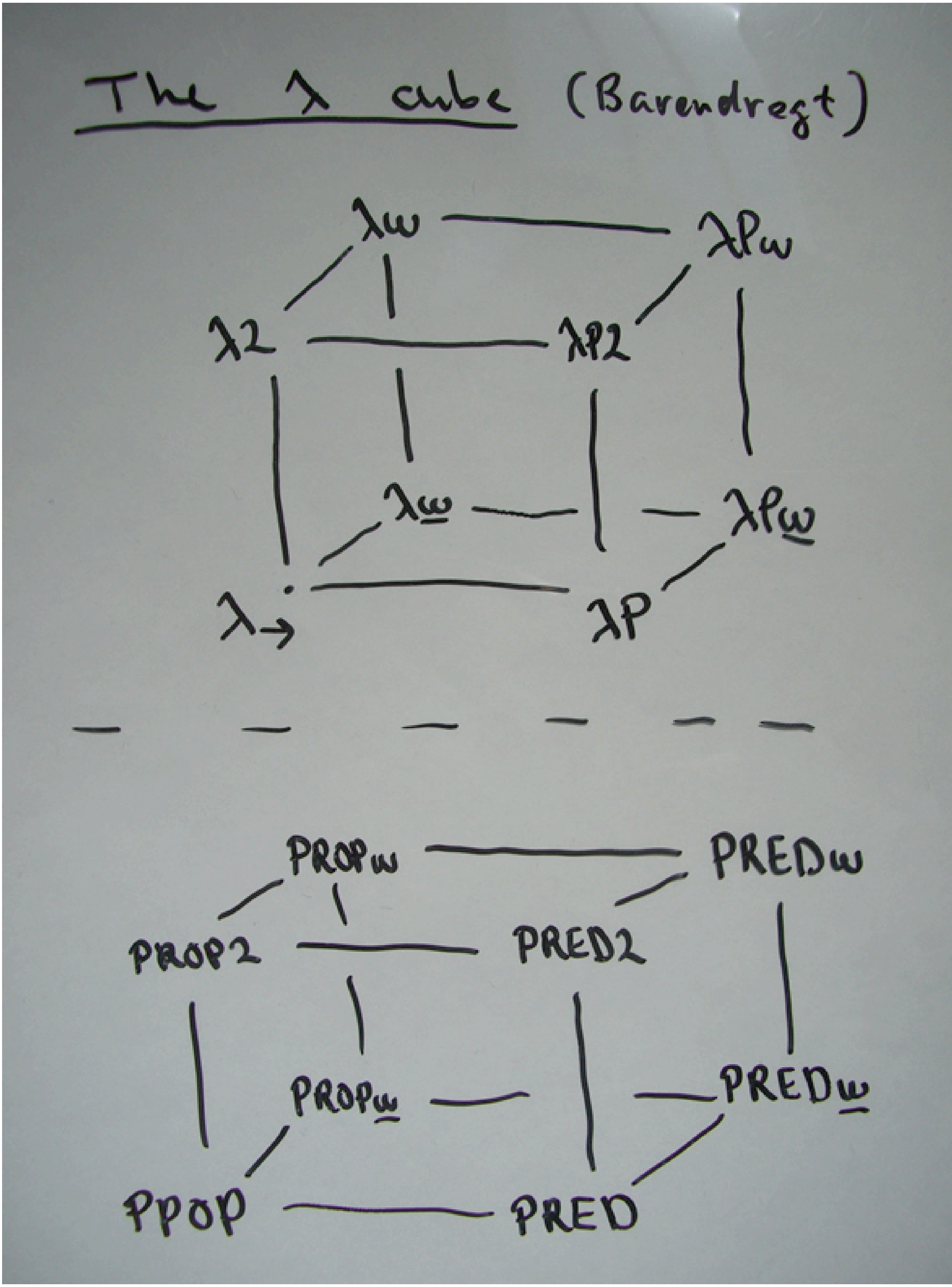
$\Gamma \vdash M : \tau$

$\Updownarrow$

there exists $\tilde{M} \in \lambda 2$ s.t. $\Gamma \vdash \tilde{M} : \tau$.

Thm Finding $\tilde{M}$ given M is undecidable. (Wells, J: 1994)

Thm (Subject reduction): If $\Gamma \vdash M : \tau$, M pure, $M \to_\beta M'$, then $\Gamma \vdash M' : \tau$


The λ cube (Barendregt)
λω
λPω
λ2
λP2
$\lambda\underline{\omega}$
$\lambda P\underline{\omega}$
λ→
λP
PROPω
PREDω
PROP2
PRED2
$PROP\underline{\omega}$
$PRED\underline{\omega}$
PPOP
PRED