This week we explore some axioms that we can add to make the theory more classical. They hold in the "standard model" of homotopy types given by simplicial sets.

Classical axioms

Here are a (non-exhaustive) list of axioms that are true classically (in the model of (discrete/ordinary) homotopy types as presented by Kan complexes in simplicial sets).

- (Law of excluded middle) $\mathrm{LEM}_{\mathcal{U}} :\equiv \prod_{(P:\mathrm{Prop}_{\mathcal{U}})}(P + \neg P)$.
- (Axiom of choice) AC says that for any *set* A, and any family of sets B over A, we have
$$\Big(\prod_{(x:A)}\|B(x)\|\Big) \to \Big\|\prod_{(x:A)}B(x)\Big\|.$$
- (Propositional resizing) $\mathrm{PRS}_{\mathcal{U},\mathcal{V}}$, for $\mathcal{V}$ containing $\mathcal{U}$, says that the map $\mathrm{Prop}_{\mathcal{U}} \to \mathrm{Prop}_{\mathcal{V}}$ is an equivalence.
- (Countable choice) CC says that AC holds for the base $A \equiv \mathbb{N}$.
- (Limited principle of omniscience) LPO says that for any sequence $\alpha : \mathbb{N} \to 2$, the proposition $\exists_{(n:\mathbb{N})}(\alpha\, n = 1)$ is decidable.
- (Markov's principle) MP says that for any sequence $\alpha : \mathbb{N} \to 2$, we have
$$\big(\neg\neg\exists_{(n:\mathbb{N})}(\alpha\, n = 1)\big) \to \exists_{(n:\mathbb{N})}(\alpha\, n = 1).$$
- (Sets cover) SC says that for any *type* A there merely exists a *set* X and a surjection $f : X \to A$.
- (Whitehead's principle) WP says that any map that is n-connected for all n is an equivalence. (We'll return to this later.)

Of course, we have some trivial implications: LEM $\to$ LPO, LPO $\to$ MP, AC $\to$ CC, and $\mathrm{LEM}_{\mathcal{V}} \to \mathrm{PRS}_{\mathcal{U},\mathcal{V}}$. It's possible to be in a semi-classical situation, where the smallest universes satisfy LEM, but larger universes do not. Here's a non-trivial implication:

Theorem 1 (Diaconescu)**.** *The axiom of choice implies the law of excluded middle.*

Proof. Consider a proposition P. Define an equivalence relation R on 2 by setting $R\,0\,1 :\equiv P$. The quotient map $q : 2 \to 2/R$ is a surjection onto a set, so by the axiom of choice, there merely exists a section $i : 2/R \to 2$. We're proving the proposition $P + \neg P$, so take such a section. Since equality on 2 is decidable, either $i(q\,0) = i(q\,1)$ or not. Since i is a section between sets, it is injective, so $(i(q\,0) = i(q\,1)) \simeq (q\,0 = q\,1) \simeq P$. Hence P or not P, as desired. □

The combination of AC and SC is also interesting:

Theorem 2. *The axiom of choice and sets cover both hold if and only if* AC *holds with no restriction on the family* B *(also known as* AC_∞*).*

Proof. From right to left: Clearly, AC_∞ implies AC. To show that sets cover, note that $A \to \|A\|_0$ is surjective with connected fibers. By AC_∞, this map merely has a section, which must be surjective.

From left to right: Given any family of types B over a set A, take a set Y with a surjection $f : Y \to \sum_{(x:A)} B(x)$ by SC. The composite $\mathrm{pr}_1 \circ f : Y \to A$ is a surjection between sets, so it has a section by AC. Compose with f to get a section of B. □

Here's an application of SC:

Theorem 3 (SC)**.** *Every category merely admits a weak equivalence functor from a strict category.*

Proof. Since we're proving a proposition, we may take a surjection $f : X_0 \to A_0$ from a set X_0 to the type of objects A_0 of A. Let $\hom_X(x, y) :\equiv \hom_A(f\,x, f\,y)$ for $x, y : X$. Then X becomes a strict category and f extends to a weak equivalence functor. □

If we further assume AC, then we may take the weak equivalence to be an adjoint equivalence.

Impredicativity

The most harmless axiom from the list, from the point of view of $(\infty, 1)$-topos models, is propositional resizing, PRS. This says that any proposition is essentially small (with respect to any universe). As a consequence, for any $A : \mathcal{U}$, the type of subsets with respect to a larger universe $\mathcal{V}$ is essentially $\mathcal{U}$-small, as long as there's a smaller universe $\mathcal{U}_0 : \mathcal{U}$. Hence, we may usefully define the type of subsets, or the powerset, of A as $\mathrm{Pow}(A) :\equiv (A \to \Omega)$, where $\Omega :\equiv \mathrm{Prop}_{\mathcal{U}_0}$ is known as the *subobject classifier*.

Propositional resizing is not, however, harmless from a logical or computational point of view: It increases the consistency strength dramatically, and it's an open problem to give a constructive model of type theory with univalence and propositional resizing (in an impredicative constructive meta-theory).

Classical sets and ordinals

For this section, we assume AC. Then $\Omega \simeq 2$. We define the type of *cardinal numbers* as $\mathrm{Card}_{\mathcal{U}} :\equiv \|\mathrm{Set}_{\mathcal{U}}\|_0$. Then we can define the usual operations of cardinal addition, multiplication, and exponentiation. Cardinal inequality is defined by the merely existence of an injection, and is then a partial order (by Cantor-Schöder-Bernstein; this just needs LEM).

We define the type of ordinals, $\mathrm{Ord}_{\mathcal{U}}$, as the type of well-ordered sets (sets with a total order such that any non-empty subset has a least element). By the SIP, this is a set. We define ordinal inequality by the (pure) existence of a *simulation* (a monotone map that doesn't skip over elements, or equivalently, an embedding of an initial segment). This is then a total order, in fact, a well-order in a larger universe: $(\mathrm{Ord}_{\mathcal{U}}, <) : \mathrm{Ord}_{\mathcal{V}}$ for $\mathcal{V}$ containing $\mathcal{U}$.

From AC it then follows that the forgetful map $\mathrm{Ord} \to \mathrm{Set}$ is surjective. For the category of sets, we can then do better than Thm 3:

Theorem 4 (AC)**.** *There is a weak equivalence functor from a strict category to the category of sets.*

Proof. As objects, take the type of ordinals (for the same universe). □

Theorem 5 (AC)**.** *The composite surjection* $\mathrm{Ord} \to \mathrm{Set} \to \mathrm{Card}$ *is split: it has a section.*

Proof. Map each cardinal to the *least* ordinal of that cardinality. □

This implies that $(\mathrm{Card}_{\mathcal{U}}, <)$ is an ordinal isomorphic to $(\mathrm{Ord}_{\mathcal{U}}, <)$, but the underlying map is not the above one. The *initial ordinals* are the image of the above section. Then the infinite initial ordinals and the infinite cardinals are both well-orders isomorphic to Ord, with ω_α and $\aleph_\alpha$ denoting the αth element of each, respectively.

Real numbers

For this section, we again assume AC for simplicity. We can define the (Dedekind) reals $\mathbb{R}$ as the type of two-sided *Dedekind cuts* (L, U). These are given by two disjoint inhabited subsets of $\mathbb{Q}$: L is the lower cut of rationals below the given real, and U is the upper cut of rationals above the given real. They are required to be *rounded* towards the real, $q \in L$ iff $\exists_{(r\in\mathbb{Q})}(q < r \wedge r \in L)$ and $q \in U$ iff

$\exists_{(r \in \mathbb{Q})}(r < q \wedge r \in U)$, and *located*, meaning $q < r \to q \in L \vee r \in U$. Then $\mathbb{R}$ is a set, and we have an embedding $\mathbb{Q} \to \mathbb{R}$ by sending $q \in \mathbb{Q}$ to $(\{r : \mathbb{Q} \mid r < q\}, \{r : \mathbb{Q} \mid q < r\})$.

The inequality relation is defined by setting $(x < y) :\equiv \exists_{(q:\mathbb{Q})}(q \in U_x \wedge q \in L_y)$ for $x \equiv (L_x, U_x)$ and $y \equiv (L_y, U_y)$.

In the usual fashion, we then prove that $\mathbb{R}$ is an ordered archimedean field that is both Dedekind- and Cauchy complete, and $\mathbb{R}$ is isomorphic to the set quotient of Cauchy sequences of rationals.

Classical analysis can then be developed in a textbook fashion, but with the added benefits of direct handles on the many naturally occurring 1- and (occasional) 2-types, etc.

Constructive analysis

Above, we've discussed the real numbers using classical axioms. The topic of *constructive analysis* is more complicated. There's an introduction in Ch. 11 of the HoTT book. There are two reasons for pursuing it despite the difficulties: (1) if we're interested in computable analysis, and (2) if we're interested in "continuous" or "smooth" versions of analysis, for example to study constructions that depend continuously on their parameters.

For (1), we can generally assume countable choice (CC) and Markov's principle (MP). Then the Cauchy reals and the Dedekind reals are still the same, and we have $\forall_{(x,y:\mathbb{R})}(x \neq y \to x < y \vee y < x)$. Cf. [2] for a development. It is *conjectured* that there is a computational model of type theory with univalence satisfying CC and MP (and SC and propositional resizing as well).

For (2), there are several settings, for instance that of the euclidean-topological homotopy types from [3] (an instance of cohesive type theory). Here, the Dedekind reals are the right continuous object, whereas the Cauchy reals behave discretely. Any function $\mathbb{R} \to \mathbb{R}$ is continuous. For other settings, such as smooth homotopy types, or smooth homotopy types with infinitesimal thickenings, etc., the Dedekind reals may not be the right object either. We then have to axiomatize a set $\mathbb{R}$ that describes the reals in the model under consideration.

Proof irrelevance and impredicativity

(This is not really specific to HoTT, but it's a piece of research news that might interest you:)

In some systems, there is a special universe of judgmentally proof irrelevant propositions: a type StrProp (of strict propositions) such that if $p, q : P$ for $P : \mathrm{StrProp}$, then $p \equiv q$.

Sometimes, as in Lean, all equality types are taken to land in StrProp (in stark conflict with univalence!). But we can imagine having another special universe StrSet of sets such that their identity types are in StrProp. Having at least $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Q}$ in StrSet would avoid many tedious transports.

The new observation by Abel and Coquand [1] is that having StrProp : StrSet and StrProp be impredicative, in the sense that StrProp is closed under Π-types, gives rise to non-normalizing terms, as is known from untyped λ-calculus. The example is very short, so check it out if it interests you!

Exercises

- (Assume AC) Show for any family of pointed and connected groupoids B over a set A that $\prod_{(x:A)} B(x)$ is again pointed and connected.
- (HoTT Book ex. 10.10) Show that AC is equivalent to the following: For every set X, there merely exists a function $f : \mathrm{Pow}_+(X) \to X$ with $f(S) \in S$ for all $S : \mathrm{Pow}_+(X)$ (the set of inhabited subsets of X).
- (HoTT Book ex. 11.6) Show that LEM implies the existence of a map $f : \mathbb{R} \to 2$ with $f(0) = 0$ and $f(x) = 1$ for $x > 0$. Show that the existence of such a map implies LPO.
- Show that a groupoid A is contractible if and only if A merely has an element that is unique up to unique isomorphism.

References

[1] Andreas Abel and Thierry Coquand. *Failure of Normalization in Impredicative Type Theory with Proof-Irrelevant Propositional Equality*. Preprint. 2019. arXiv: 1911.08174.

[2] Errett Bishop and Douglas Bridges. *Constructive analysis*. Vol. 279. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer-Verlag, Berlin, 1985, pp. xii+477. ISBN: 3-540-15066-8. DOI: 10.1007/978-3-642-61667-9.

[3] Michael Shulman. "Brouwer's fixed-point theorem in real-cohesive homotopy type theory". In: *Mathematical Structures in Computer Science* (2017), pp. 1–86. DOI: 10.1017/S0960129517000147.