Transfinitely iterated fixpoint theories

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Outline

Introduction

We summarize the paper by Jäger, Kahle, Setzer, and Strahm: *The proof-theoretic analysis of transfinitely iterated fixed point theories* [JKSS99].

Our goal is to determine the proof-theoretical ordinals of a family of theories, $\widehat{\mathsf{ID}}_\alpha$. Part I will sketch the lower bound. Part II will concern the upper bound.

Ordinals

References: [Veb08, Sch77, Poh09]

In proof theory we use ordinals to measure infinitary objects. Ordinals are equivalence classes of well-orderings. If α and β are ordinals, then $\alpha < \beta$ if (a representative of) α is isomorphic to a proper initial segment of (a representative of) β. This relation well-orders the class On of all ordinals. Therefore, we identify an ordinal with its canonical representative:

$$\alpha = \{ \xi \mid \xi < \alpha \}$$

In this way, ordinals become transitive sets that are well-ordered by $\in$ (von Neumann ordinals).

Order-types and enumerations

Every well-founded recursive relation $\prec$ on ω determines an ordinal less than ω_1^{CK}. Kleene noted that every recursive ordinal can be represented by a primitive recursive (even Kalmár elementary) well-order on ω. [Kle55]

Any well-order $\prec$ is order-isomorphic to its representing von Neumann ordinal, $\mathrm{otyp}(\prec) \in \mathsf{On}$, through an *enumerating* function:

$$\mathrm{en}_{\prec} \colon \mathrm{otyp}(\prec) \to \mathrm{field}(\prec)$$

Every subclass $M \subset \mathsf{On}$ is well-ordered with either

$\mathrm{otyp}(M) = \mathsf{On}$ (if M is a proper class), or

$\mathrm{otyp}(M) \in \mathsf{On}$ (if M is a set).

Normal functions and club classes

Let Ω be the least uncountable ordinal.

We say that $M \subset \mathrm{On}$ is *unbounded* in Ω if for each $\alpha < \Omega$ there is is $\beta \in M \cap \Omega$ with $\alpha < \beta$.
We say that $M \subset \mathrm{On}$ is *closed* in Ω if for every countable $A \subset M$ we have $\sup(A) \in M$.
An order-preserving function $f\colon \mathrm{On} \dashrightarrow \mathrm{On}$ is called *normal* on Ω if $\mathrm{dom}(f) \supset \Omega$ and f is continuous.

Lemma

A class $M \subset \mathrm{On}$ is club in Ω iff en_M is normal on Ω.

(Analogues of the above exist for each regular cardinal $\kappa \geq \Omega$.)

Ordinal Arithmetic

Given an ordinal $\alpha < \Omega$, the class $\mathsf{On}_\alpha := \{\, \beta \in \mathsf{On} \mid \alpha \leq \beta \,\}$ is club, so the enumerating function is normal on Ω. Its value on ξ is denoted $\alpha + \xi$ and is called the *ordinal sum* of α and ξ.

Note that $\alpha + \xi$ can also be defined by transfinite recursion on ξ.

Let $\mathbb{H} := \{\, \alpha \in \mathsf{On} \mid \alpha \neq 0 \wedge \forall \xi, \eta < \alpha.\, \xi + \eta < \alpha \,\}$. We call ordinals in $\mathbb{H}$ *additively indecomposable* or *principal*.

Lemma

$\mathbb{H}$ *is club in* Ω, *so* $\mathrm{en}_{\mathbb{H}}$ *is normal. We write* $\omega^\xi := \mathrm{en}_{\mathbb{H}}(\xi)$.

Cantor Normal Form

Theorem (Cantor Normal Form)

For all ordinals $\alpha \neq 0$, there are uniquely determined principal ordinals $\alpha_1, \ldots, \alpha_n$ such that

$$\alpha = \alpha_1 + \cdots + \alpha_n \quad \textit{and} \quad \alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n.$$

Thus, noting that powers of ω enumerate the principal ordinals, and collecting equals terms, we get the form

$$\alpha = \omega^{\beta_1} k_1 + \cdots + \omega^{\beta_n} k_n \quad \text{where} \quad \beta_1 > \beta_2 \cdots > \beta_n.$$

The Cantor Normal Form can be used to form a notation system for ordinals less than ε_0.

Ordinal class differentiation

For $f\colon \mathrm{On} \dashrightarrow \mathrm{On}$ put $\mathrm{Fix}(f) := \{\, \eta \in \mathrm{dom}(f) \mid f(\eta) = \eta \,\}$. The *derivative* of f is defined by

$$f' := \mathrm{en}_{\mathrm{Fix}(f)}$$

We define the derivative of a class $M \subset \mathrm{On}$ as $M' := \mathrm{Fix}(\mathrm{en}_M)$.
Facts:

- If f is normal, then $\mathrm{Fix}(f)$ is club, and so f' is again normal.
- If M is club, then M' is again club.
- A countable intersection of club classes is again club.

The Veblen Hierarchy

The Veblen Hierarchy of critical ordinals is defined by

$$\begin{aligned} \mathsf{Cr}(0) &:= \mathbb{H} \\ \mathsf{Cr}(\alpha+1) &:= \mathsf{Cr}(\alpha)' \\ \mathsf{Cr}(\lambda) &:= \bigcap_{\xi<\lambda} \mathsf{Cr}(\xi). \end{aligned}$$

Then we put $\varphi_\alpha := \mathrm{en}_{\mathsf{Cr}(\alpha)}$. We also write $\varphi(\alpha,\beta) = \varphi_\alpha(\beta)$. Then

$$\varphi(0,\beta) = \omega^\beta \quad \text{and} \quad \varphi(1,\beta) = \varepsilon_\beta$$

Theorem

For $\alpha < \Omega$, the class $\mathsf{Cr}(\alpha)$ is club in Ω, so φ_α is normal on Ω.

Order relations

Theorem

We have:

(A) $\varphi_{\alpha_1}(\beta_1) = \varphi_{\alpha_2}(\beta_2)$ *iff either of the following:*

- (i) $\alpha_1 < \alpha_2 \wedge \beta_1 = \varphi_{\alpha_2}(\beta_2)$
- (ii) $\alpha_1 = \alpha_2 \wedge \beta_1 = \beta_2$
- (iii) $\alpha_1 > \alpha_2 \wedge \varphi_{\alpha_1}(\beta_1) = \beta_2$

(B) $\varphi_{\alpha_1}(\beta_1) < \varphi_{\alpha_2}(\beta_2)$ *iff either of the following:*

- (i) $\alpha_1 < \alpha_2 \wedge \beta_1 < \varphi_{\alpha_2}(\beta_2)$
- (ii) $\alpha_1 = \alpha_2 \wedge \beta_1 < \beta_2$
- (iii) $\alpha_1 > \alpha_2 \wedge \varphi_{\alpha_1}(\beta_1) < \beta_2$

This forms the basis for a notation system for ordinals less than Γ_0.

The Small Veblen Ordinal

Define $\mathsf{Cr}_\xi(\alpha)$ by

$$\begin{aligned}
\mathsf{Cr}_0(0) &:= \mathbb{H} \\
\mathsf{Cr}_\xi(\alpha+1) &:= \mathsf{Cr}_\xi(\alpha)' \\
\mathsf{Cr}_\xi(\lambda) &:= \cap_{\eta<\lambda}\mathsf{Cr}_\xi(\eta) \\
\mathsf{Cr}_{\alpha+1}(0) &:= \left\{\, \xi < \Omega \;\middle|\; \xi \in \cap_{\lambda<\xi}\mathsf{Cr}_\alpha(\lambda) \,\right\} \\
\mathsf{Cr}_\lambda(0) &:= \cap_{\xi<\lambda}\mathsf{Cr}_\xi(0)
\end{aligned}$$

and put $\varphi_{\xi,\alpha} := \mathrm{en}_{\mathsf{Cr}_\xi(\alpha)}$. Continuing this line of reasoning, we get n-ary φ-functions $\Omega^n \to \Omega$ for each $n < \omega$. Letting Φ_0 be the least ordinal not expressible with these, we also get an elementary notation system $\prec$ on ω for the ordinals less than Φ_0.

The theories $\widehat{\mathsf{ID}}_\alpha$

We define the systems $\widehat{\mathsf{ID}}_\alpha$ for $\alpha < \Phi_0$. Let $\mathcal{L}$ be the language of first-order arithmetic. Terms are built from variables by means of primitive recursive functions. The predicates are the primitive recursive relations, as well as a unary predicate U.

Inductive operator forms

If P and Q are fresh unary predicates, then we let $\mathcal{L}(P, Q)$ denote the extension of $\mathcal{L}$ with P and Q. We let $\mathcal{A}(P, Q, x, y)$ range over formulas of $\mathcal{L}(P, Q)$ that

(1) are P-*positive* (only positive literals $P(t)$ occur), and

(2) have at most x and y free.

Such formulas are called *inductive operator forms*.

Transfinite induction

Let $\sqsubset$ be a primitive recursive relation, and let $A(x)$ be a formula with a distinguished variable. Let s be a term. Then we set:

$$\begin{aligned} A(\sqsubset x) &:= \forall y \sqsubset x.\, A(y) \\ \mathrm{Prog}(\sqsubset, A) &:= \forall x.\, A(\sqsubset x) \to A(x) \\ \mathrm{TI}(\sqsubset, A) &:= \mathrm{Prog}(\sqsubset, A) \to \forall x.\, A(x) \\ \mathrm{TI}(\sqsubset, s, A) &:= \mathrm{Prog}(\sqsubset, A) \to A(\sqsubset s) \end{aligned}$$

If we just write $\mathrm{Prog}(A)$, $\mathrm{TI}(A)$, or $\mathrm{TI}(s, A)$, it is understood that we take $\sqsubset$ to be $\prec$.

The language of $\widehat{\mathsf{ID}}_\alpha$

For each $\alpha < \Phi_0$ we have the theory $\widehat{\mathsf{ID}}_\alpha$ of α times iterated fixpoints. The language is $\mathcal{L}_{\mathsf{fix}}$, which we obtain from $\mathcal{L}$ by adding a new unary predicate symbol $P^{\mathcal{A}}$ for each inductive operator form $\mathcal{A}(P, Q, x, y)$.
We define:

$$P_s^{\mathcal{A}}(u) := P^{\mathcal{A}}(\langle u, s\rangle)$$
$$P_{\prec a}^{\mathcal{A}}(t) := \big(t = \langle u, s\rangle\big) \wedge s \prec a \wedge P_s^{\mathcal{A}}(u)$$

The intended meaning is that $P_{\prec a}^{\mathcal{A}} = \sum_{s \prec a} P_s^{\mathcal{A}}$.

Axioms of $\widehat{\mathsf{ID}}_\alpha$

The axioms of $\widehat{\mathsf{ID}}_\alpha$ comprise

(i) those of Peano arithmetic with the induction scheme for all $\mathcal{L}_{\mathsf{fix}}$-formulas,

(ii) fixpoint axiom: for each inductive operator form $\mathcal{A}(P, Q, x, y)$ we have

$$\forall a \prec \alpha. \forall x. P_a^{\mathcal{A}}(x) \leftrightarrow \mathcal{A}(P_a^{\mathcal{A}}, P_{\prec a}^{\mathcal{A}}, x, a),$$

(iii) the axioms $\mathrm{TI}(\alpha, A)$ for each $\mathcal{L}_{\mathsf{fix}}$-formula A.

Previous Results

The theories $\widehat{\mathsf{ID}}_n$ were first introduced by Feferman [Fef82]. It was established that the proof-theoretical ordinal of $\widehat{\mathsf{ID}}_n$ is α_n, where $\alpha_0 := \varepsilon_0$, and $\alpha_{n+1} := \varphi\alpha_n 0$.
Therefore, the union, $\widehat{\mathsf{ID}}_{<\omega}$, has proof-theoretical ordinal Γ_0, the first predicatively inaccessible ordinal.
JKSS extends these results to $\widehat{\mathsf{ID}}_\alpha$ for $\alpha \geq \omega$.

Main Theorem of JKSS

Fix an ordinal $\alpha < \Phi_0$. Let $\varepsilon(\alpha)$ denote the least ε-number greater than α. Define inductively

$$(\alpha \mid 0) := \varepsilon(\alpha), \qquad (\alpha \mid m+1) := \varphi(\alpha \mid m)0.$$

Theorem (Main Theorem)

Let α have Cantor Normal Form

$$\alpha = \omega^{1+\alpha_n} + \cdots + \omega^{1+\alpha_1} + m,$$

where $\alpha_n \geq \cdots \geq \alpha_1$ and $m < \omega$. Then the proof-theoretic ordinal of $\widehat{\mathsf{ID}}_\alpha$ is

$$|\widehat{\mathsf{ID}}_\alpha| = \varphi 1 \alpha_n(\varphi 1 \alpha_{n-1}(\ldots \varphi 1 \alpha_1(\alpha \mid m)) \ldots).$$

Lower Bounds: Ordinal Analysis of PA

Let us recall how to show that $|\mathsf{PA}| \geq \varepsilon_0$.
Since $\varepsilon_0 = \sup\{\exp_\omega^n(0) \mid n < \omega\}$, this is done by showing:

$$\text{if } \mathsf{PA} \vdash \mathrm{TI}(\alpha, U) \text{ then } \mathsf{PA} \vdash \mathrm{TI}(\omega^\alpha, U)$$

A basic ingredient is the substitution lemma:

Lemma

Let $F(U)$ and $G(x)$ be formulas in $\mathcal{L}(U)$. If $\mathsf{PA} \vdash F(U)$*, then* $\mathsf{PA} \vdash F(\lambda x.\, G(x))$.

Lower bound of PA, continued

Define the jump of X by

$$\mathcal{J}(X) := \lambda\alpha.\,\forall\xi.\,X(\prec \xi) \to X(\prec \xi + \omega^{\alpha})$$

Then one shows first:

Lemma

PA $\vdash \mathrm{Prog}(X) \to \mathrm{Prog}(\mathcal{J}(X))$.

And then:

Lemma

PA $\vdash \mathrm{TI}(\alpha, \mathcal{J}(X)) \to \mathrm{TI}(\omega^{\alpha}, X)$.

Combining this with substitution, gives the lower bound for PA.

Theories for self-reflecting truth

To use the well-ordering techniques for predicative systems, we will study theories for self-reflecting truth. We will see that these embed into the $\widehat{\mathrm{ID}}_\alpha$, where they will facilitate our well-ordering proofs.

Kripke [Kri75] gave in his "Outline of a theory of truth" a simultaneous inductive definition of predicates T and F over a given language, which give partial truth and falsity predicates grounded in atomic formulas.

Here, we iterate this procedure. We use Quine quotes, $\ulcorner A \urcorner$, to mean terms for Gödel numbers.

Theories for self-reflecting truth, language

The language $\mathcal{L}_{\mathsf{srt}}$ extends $\mathcal{L}$ by two binary relation symbols: T and F.

$T_s(t)$ means that t is the code of a true formula at level s.

$F_s(t)$ means that t is the code of a false formula at level s.

The sublanguage $\mathcal{L}^{\alpha}_{\mathsf{srt}}$ is obtained by restricting levels s in atoms $T_s(t)$ and $F_s(t)$ to closed terms with $s \preceq \alpha$.

Gödelization of $\mathcal{L}_{\mathsf{srt}}^{\alpha}$

We Gödelize the languages $\mathcal{L}_{\mathsf{srt}}^{\alpha}$ uniformly in $\alpha < \Phi_0$ to get Gödel numbers $\ulcorner t \urcorner$ and $\ulcorner A \urcorner$ for each $\mathcal{L}$-term t and each $\mathcal{L}_{\mathsf{srt}}^{\alpha}$-formula A.

We have the following primitive recursive functions and predicates:

$\mathrm{CTer}(x)$ – x codes a closed term of $\mathcal{L}$.

$\mathrm{For}_n(f, a)$ – f codes an $\mathcal{L}_{\mathsf{srt}}^{a}$-formula with at most n free variables.

$\mathrm{Atm}(f, a)$ – f codes a positive literal of $\mathcal{L}_{\mathsf{srt}}^{a}$.

$\mathrm{num}(x)$ – the code of the xth numeral of $\mathcal{L}$.

$\mathrm{val}(z)$ – the value of the closed term z.

$\mathrm{neg}(f)$ – the negation of the atom f.

$\mathrm{and}(f, g)$ – the conjunction of f and g (also have $\mathrm{or}(f, g)$).

$\mathrm{all}(x, f)$ – universal quantification of f with respect to the xth variable (also have $\mathrm{ext}(f, g)$).

Gödelization of $\mathcal{L}^{\alpha}_{\mathsf{srt}}$, continued

We write $\mathrm{Sen}(f,a)$ instead of $\mathrm{For}_0(f,a)$ and $\dot{x}$ instead of $\mathrm{num}(x)$.
If $\mathrm{For}_n(f,a)$ and $\mathrm{CTer}(x_1),\dots,\mathrm{CTer}(x_n)$, then $f(x_1,\dots,x_n)$ denotes the code of the formula obtained from f by simultaneuosly substituting the ith variable of f with x_i for $i=1,\dots,n$.
Similarly, if A is an $\mathcal{L}^{\alpha}_{\mathsf{srt}}$-formula with at most n free variables, then

$$\ulcorner A(\dot{x}_1,\dots,\dot{x}_n)\urcorner \text{ is an abbreviation for } \ulcorner A\urcorner(\dot{x}_1,\dots,\dot{x}_n),$$

and if we furthermore have $\mathrm{CTer}(x_1),\dots,\mathrm{CTer}(x_n)$, then

$$\ulcorner A(x_1,\dots,x_n)\urcorner \text{ is an abbreviation for } \ulcorner A\urcorner(x_1,\dots,x_n).$$

If furthermore R is an n-ary relation symbol of $\mathcal{L}$, then

$$\ulcorner R(x_1,\dots,x_n)\urcorner \text{ is the code of the corresponding atom.}$$

Lastly, if $\mathrm{CTer}(a)$ and $\mathrm{CTer}(x)$, then

$$\ulcorner T_a(x)\urcorner \text{ and } \ulcorner F_a(x)\urcorner \text{ are codes of the corresponding atoms.}$$

Theories for self-reflecting truth, axiomatization

Let α be an ordinal less than Φ_0.
The language of the system SRT_α for α *times iterated self-reflecting truth* is the *unrestricted* $\mathcal{L}_{\mathsf{srt}}$. The theory comprises:

(i) the axioms of Peano arithmetic with the induction scheme for all $\mathcal{L}_{\mathsf{srt}}$-formulas.

(ii) the axioms $\mathrm{TI}(\alpha, A)$ for all $\mathcal{L}_{\mathsf{srt}}$-formulas A.

(iii) the axiom groups I to III on the following slides.

SRT$_\alpha$ axiom group I – atomic truth

(1) For each n-ary relation symbol R of $\mathcal{L}$:

$$\begin{aligned} &\mathrm{CTer}(x_1) \wedge \cdots \wedge \mathrm{CTer}(x_n) \wedge a \prec \alpha \\ &\qquad \to \Big(T_a(\ulcorner R(x_1, \ldots, x_n)\urcorner) \leftrightarrow R(\mathrm{val}(x_1), \ldots, \mathrm{val}(x_n))\Big) \\ &\qquad\qquad \wedge \Big(F_a(\ulcorner R(x_1, \ldots, x_n)\urcorner) \leftrightarrow \neg R(\mathrm{val}(x_1), \ldots, \mathrm{val}(x_n))\Big) \end{aligned}$$

(2)

$$\begin{aligned} &\mathrm{CTer}(x) \wedge \mathrm{CTer}(b) \wedge \mathrm{val}(b) \prec a \prec \alpha \\ &\to \Big(T_a(\ulcorner T_b(x)\urcorner) \leftrightarrow T_{\mathrm{val}\, b}(\mathrm{val}(x))\Big) \wedge \Big(T_a(\ulcorner F_b(x)\urcorner) \leftrightarrow F_{\mathrm{val}\, b}(\mathrm{val}(x))\Big) \\ &\wedge \Big(F_a(\ulcorner T_b(x)\urcorner) \leftrightarrow \neg T_{\mathrm{val}\, b}(\mathrm{val}(x))\Big) \wedge \Big(F_a(\ulcorner F_b(x)\urcorner) \leftrightarrow \neg F_{\mathrm{val}\, b}(\mathrm{val}(x))\Big) \end{aligned}$$

SRT$_\alpha$ axiom group II – composed truth

(3) $\mathrm{Atm}(f,a) \wedge a \prec \alpha \to \big(T_a(\mathrm{neg}(f)) \leftrightarrow F_a(f)\big) \wedge \big(F_a(\mathrm{neg}(f)) \leftrightarrow T_a(f)\big)$.

(4) $\mathrm{Sen}(f,a) \wedge \mathrm{Sen}(g,a) \wedge a \prec \alpha$
$$\to \big(T_a(\mathrm{and}(f,g)) \leftrightarrow T_a(f) \wedge T_a(g)\big)$$
$$\wedge \big(F_a(\mathrm{and}(f,g)) \leftrightarrow F_a(f) \vee F_a(g)\big).$$

(5) dual axiom for disjunction

(6) $\mathrm{Sen}(\mathrm{all}(v,f),a) \wedge a \prec \alpha$
$$\to \big(T_a(\mathrm{all}(v,f)) \leftrightarrow (\forall x) T_a(f(\dot{x}))\big)$$
$$\wedge \big(F_a(\mathrm{all}(v,f)) \leftrightarrow (\exists x) F_a(f(\dot{x}))\big).$$

(7) dual axiom for existential quantification.

SRT$_\alpha$ axiom group III – self-reflecting truth

(8)

$$
\begin{aligned}
&\mathrm{CTer}(x) \wedge \mathrm{CTer}(a) \wedge \mathrm{val}(a) \prec \alpha \\
&\qquad \to \Big(T_{\mathrm{val}(a)}(\ulcorner T_a(x) \urcorner) \leftrightarrow T_{\mathrm{val}(a)}(\mathrm{val}(x))\Big) \\
&\qquad \wedge \Big(T_{\mathrm{val}(a)}(\ulcorner F_a(x) \urcorner) \leftrightarrow F_{\mathrm{val}(a)}(\mathrm{val}(x))\Big) \\
&\qquad \wedge \Big(F_{\mathrm{val}(a)}(\ulcorner T_a(x) \urcorner) \leftrightarrow T_{\mathrm{val}(a)}(\mathrm{val}(x))\Big) \\
&\qquad\qquad \wedge \Big(F_{\mathrm{val}(a)}(\ulcorner F_a(x) \urcorner) \leftrightarrow T_{\mathrm{val}(a)}(\mathrm{val}(x))\Big)
\end{aligned}
$$

Embedding SRT_α into $\widehat{\mathsf{ID}}_\alpha$

We follow Sol's paper *Reflecting on incompleteness* [Fef91].

Fix $\alpha < \Phi_0$. We want to model SRT_α in $\widehat{\mathsf{ID}}_\alpha$. The idea is to interpret T and F by a fixpoint family Q so that

$$T_a(t) \leftrightarrow Q_a(\langle t, 0\rangle) \qquad \text{and} \qquad F_a(t) \leftrightarrow Q_a(\langle t, 1\rangle).$$

We need to find a suitable inductive operator form $\mathcal{A}(Q_a, Q_{\prec a}, x, a)$. We give an informal definition of this formula on the following slides.

Clauses for truth fixpoint family, I

$$\begin{aligned}
\mathcal{A}(Q_a, Q_{\prec a}, x, a) := \text{let } x = \langle t, y\rangle \text{ in: case of} \\
&(1)\ t = \ulcorner R(x_1, \ldots, x_n)\urcorner \text{ where} \\
&\qquad \mathrm{CTer}(x_1) \wedge \cdots \wedge \mathrm{CTer}(x_n) : \\
&\quad (R(val(x_1), \ldots, val(x_n)) \wedge y = 0) \vee \\
&\quad (\neg R(val(x_n), \ldots, val(x_n)) \wedge y = 1) \\
&(2)\ t = \ulcorner T_b(f)\urcorner \text{ where} \\
&\qquad \mathrm{CTer}(f) \wedge \mathrm{CTer}(b) \wedge \mathrm{val}(b) \prec a : \\
&\quad (Q_{\mathrm{val}(b)}(\langle \mathrm{val}\, f, 0\rangle) \wedge y = 0) \vee \\
&\quad (Q_{\mathrm{val}(b)}(\langle \mathrm{val}\, f, 1\rangle) \wedge y = 1)
\end{aligned}$$

$\ldots$

Clauses for truth fixpoint family, II

...

(2') $t = \ulcorner F_b(f) \urcorner$ where

$$\mathrm{CTer}(f) \wedge \mathrm{CTer}(b) \wedge \mathrm{val}(b) \prec a :$$

$$(Q_{\mathrm{val}(b)}(\langle \mathrm{val}\, f, 1\rangle) \wedge y = 0) \vee$$

$$(Q_{\mathrm{val}(b)}(\langle \mathrm{val}\, f, 0\rangle) \wedge y = 1)$$

(3) $t = \mathrm{neg}(f)$ where $\mathrm{Atm}(f, a)$:

$$(Q_a(\langle f, 1\rangle) \wedge y = 0) \vee (Q_a(\langle f, 0\rangle) \wedge y = 1)$$

...

(the other conditions (4)–(7) for composed truth are similar)

...

Clauses for truth fixpoint family, III

...

(8) $t = \ulcorner T_{\dot{a}}(f) \urcorner$ where $\mathrm{CTer}(f)$:

$$(Q_a(\langle \mathrm{val}\, f, 0\rangle) \wedge y = 0) \vee$$
$$(Q_a(\langle \mathrm{val}\, f, 1\rangle) \wedge y = 1)$$

(8') $t = \ulcorner F_{\dot{a}}(f) \urcorner$ where $\mathrm{CTer}(f)$:

$$(Q_a(\langle \mathrm{val}\, f, 1\rangle) \wedge y = 0) \vee$$
$$(Q_a(\langle \mathrm{val}\, f, 0\rangle) \wedge y = 1)$$

Ramified sets in SRT_α

The systems SRT_α has a natural notion of ramified subsets of ω, conceived of as propositional functions. We set:

$$f \in S_a := \mathrm{For}_1(f, a) \wedge \forall x.\, T_a(f(\dot{x})) \leftrightarrow \neg F_a(f(\dot{x}))$$

– meaning f is a set of level a.

$$x \in_a f := T_a(f(\dot{x}))$$

– meaning x is an element of the set f.

Well-ordering proofs for SRT_α

We consider the notion of having transfinite induction up to a for all sets of level less than c, formalized as:

$$I^c(a) := \forall b \prec c.\, \forall f \in S_b.\ \mathrm{TI}(a, f).$$

Here $\mathrm{TI}(a, f)$ means $\mathrm{TI}(a, \lambda x.\, x \in_b f)$, which we recall is

$$\Big(\forall x.\, (\forall y.\, y \prec x \to y \in_b f) \to x \in_b f\Big) \to \forall x \prec a.\, x \in_b f.$$

Lemma

SRT_α *proves:*

$$\forall \ell.\ \mathrm{Lim}(\ell) \wedge \ell \preceq \alpha \to \exists f \in S_\ell.\, \forall a.\, I^\ell(a) \leftrightarrow a \in_\ell f.$$

Well-ordering proofs for SRT$_\alpha$, continued

The first step is from predicative proof theory:

Lemma

SRT$_\alpha$ *proves:*

$$\forall \ell, a.\ \mathrm{Lim}(\ell) \wedge \ell \preceq \alpha \wedge I^{\ell}(a) \rightarrow I^{\ell}(\varphi a 0).$$

This is done by introducing and using the Veblen-a-jumps:

$$\mathcal{V}\mathcal{J}_a(\ell) := \lambda b.\, I^{\ell}(b) \rightarrow I^{\ell}(\varphi a b).$$

Note: This step is easier for studying SRT$_\alpha$ (and so $\widehat{\mathrm{ID}}_\alpha$) for $\alpha \geq \omega$. For finite α, we need more delicate analysis!

Well-ordering proofs for SRT_α, continued

We see that these well-ordering proofs rely on room to maneuver in the levels. So we define:

$$a \uparrow b := \exists c, \ell.\ \mathrm{Lim}(\ell) \land b = c + a \cdot \ell$$

This is enough to formulate the main lemma:

Lemma (Main Lemma I)

Let $\mathrm{Main}_\alpha(a)$ *be defined as:*

$$\mathrm{Main}_\alpha(a) := \forall b, c.\ c \preceq \alpha \land \omega^{1+a} \uparrow c \land I^c(b) \to I^c(\varphi 1ab)$$

Then SRT_α *proves* $\mathrm{Prog}(\lambda a.\ \mathrm{Main}_\alpha(a))$.

Well-ordering proofs for SRT_α, completed

Corollary

SRT_α *proves:*

$$\forall c, c_0, d.\, c \preceq \alpha \wedge c = c_0 + \omega^{1+d} \to \mathrm{Prog}(\lambda e.\, I^c(\varphi 1 de)).$$

The whole well-ordering proof starts from our analysis of number theory:

Lemma

Let $\beta < \varepsilon(\alpha)$ and A be an formula of $\mathcal{L}_{\mathsf{srt}}$. Then SRT_α *proves* $\mathrm{TI}(\beta, A)$.

Combined, these results give the lower bound of the main theorem.

Upper bounds

We will sketchs the cut-elimination arguments used to establish the upper bound in the main theorem.
For this, we will employ subsystems H_α of a semi-formal system H_∞.
To motivate this, we will briefly recall the goals and the framework of ordinal analysis.

Axioms for number theory

Recall one motivation for proof-theoretical analysis: the desire to quantify how well an axiom system approximates the set of true number theoretical pseudo-Π^1_1-sentences (those containing free set variables, but no quantification over set variables).
For an axiom system T, we define the Π^1_1-ordinal of T by:

$$|T|_{\Pi^1_1} = \sup\{\, |\varphi| \mid T \vdash \varphi \,\}.$$

using some measure of complexity, $|\varphi|$, of pseudo-Π^1_1-sentences.
A nice measure is the *truth-complexity*, $\mathrm{tc}(\varphi)$, defined using a semi-formal system.
It turns that for most theories we can restrict ourselves to the special sentences, $\mathrm{TI}(\sqsubset, U)$.

Proof-theoretical ordinals

Let T be a theory. The proof-theoretical ordinal of T, denoted $|T|$ is defined by:

$$|T| := \sup\{\, \mathrm{otyp}(\sqsubset) \mid \sqsubset \text{ primitive recursive and } T \vdash \mathrm{TI}(\sqsubset, U) \,\}$$

For theories with proof-theoretical ordinal less than Φ_0, we have

$$|T| = \sup\{\, \alpha \mid T \vdash \mathrm{TI}(\alpha, U) \,\}$$

The key to determining proof-theoretical ordinals is the following:

Theorem (Boundedness Theorem)

Let $\prec$ be a primitive recursive, binary, transitive, well-founded relation. If $\mathrm{Lim}(\mathrm{otyp}(\prec))$, then $\mathrm{otyp}(\prec) = \mathrm{tc}(\mathrm{TI}(\prec))$.

Aczel's trick

Now to fixed point theories:
To model one fixpoint, we can use Aczel's trick:

Lemma

Given an operator form, $A(P, x)$, there's a Σ^1_1-formula $\tilde{P}(x)$ such that

$$\Sigma^1_1\text{-AC} \vdash A(\tilde{P}, x) \leftrightarrow \tilde{P}(x)$$

Proof.

Let $E(z, x, y)$ be a Σ^1_1-formula which provably enumerates all Σ^1_1-formulas $Q(x, y)$ as $z = 0, 1, 2, \ldots$. Consider $A(\lambda u.\, E(z, z, u), x)$. This formula is provably equivalent to some $Q_e(z, x)$ in Σ^1_1-AC, so take $\tilde{P} := Q_e(e, x)$. □

But this is not the route taken in the present paper.

The semi-formal system

We formulate a Tait-style system H_∞. It is formulated in the language $\mathcal{L}_\infty$ which extends $\mathcal{L}$ by unary relation symbols $P^{\mathcal{A}}_{\beta}$ and $P^{\mathcal{A}}_{<\beta}$ for each inductive operator form $\mathcal{A}$ and each ordinal $\beta < \Phi_0$. The *preformulas* of $\mathcal{L}_\infty$ are generated inductively by:

1. Every literal of $\mathcal{L}$ is an $\mathcal{L}_\infty$ preformula.
2. If t is a number term, then the literals $P^{\mathcal{A}}_{\beta}(t)$, $P^{\mathcal{A}}_{<\beta}(t)$, $\neg P^{\mathcal{A}}_{\beta}(t)$, and $\neg P^{\mathcal{A}}_{<\beta}(t)$ are $\mathcal{L}_\infty$ preformulas.
3. $\mathcal{L}_\infty$ preformulas are closed under conjunction, disjunction, and universal and existential quantification.

We then define the $\mathcal{L}_\infty$ *formulas* to be the *closed* preformulas.
We detail the axioms and rules of inferences on the following slides.

The semi-formal system, axioms

I. Axioms, group 1 For all Γ, numerically equivalent literals A and B, and all true literals C:

$$\Gamma, \neg A, B \quad \text{and} \quad \Gamma, C.$$

II. Axioms, group 2 For all Γ, all closed terms s with $\mathrm{pair}(s)$ false, and all closed terms t with $\mathrm{pair}(t)$ and $\beta \leq |(t)_1|$ true:

$$\Gamma, \neg P^{\mathcal{A}}_{<\beta}(s) \quad \text{and} \quad \Gamma, \neg P^{\mathcal{A}}_{<\beta}(t).$$

The semi-formal system, fixed point rules

III. Fixed point rules, group 1 For all Γ and all closed terms t so that $\mathrm{pair}(t)$ and $|(t)_1| = \alpha < \beta$:

$$\frac{\Gamma, P^{\mathcal{A}}_{\alpha}((t)_0)}{\Gamma, P^{\mathcal{A}}_{<\beta}(t)}, \quad \text{and} \quad \frac{\Gamma, \neg P^{\mathcal{A}}_{\alpha}((t)_0)}{\Gamma, \neg P^{\mathcal{A}}_{<\beta}(t)}$$

IV. Fixed point rules, group 2 For all Γ, all closed terms s, and all closed terms t with $|t| = \beta$:

$$\frac{\Gamma, \mathcal{A}(P^{\mathcal{A}}_{\beta}, P^{\mathcal{A}}_{<\beta}, s, t)}{\Gamma, P^{\mathcal{A}}_{\beta}(s)}, \quad \text{and} \quad \frac{\Gamma, \neg\mathcal{A}(P^{\mathcal{A}}_{\beta}, P^{\mathcal{A}}_{<\beta}, s, t)}{\Gamma, \neg P^{\mathcal{A}}_{\beta}(s)}$$

The semi-formal system, remaining rules

V. Propositional rules For all Γ and all $\mathcal{L}_\infty$ formulas A and B:

$$\frac{\Gamma, A}{\Gamma, A \vee B}, \quad \text{and} \quad \frac{\Gamma, B}{\Gamma, A \vee B}, \quad \text{and} \quad \frac{\Gamma, A \qquad \Gamma, B}{\Gamma, A \wedge B}$$

VI. Quantifier rules For all Γ, all $\mathcal{L}_\infty$ preformulas $A(x)$ with only x free, and all closed terms s:

$$\frac{\Gamma, A(x)}{\Gamma, \exists x.\, A(x)} \quad \text{and} \quad \frac{\Gamma, A(t) \text{ for all closed terms } t}{\Gamma, \forall x.\, A(x)}\,(\omega)$$

VII. Cut rule For all Γ, and all $\mathcal{L}_\infty$ formulas A:

$$\frac{\Gamma, A \qquad \Gamma, \neg A}{\Gamma}$$

Partial cut elimination

The language $\mathcal{L}_\alpha$ is the sublanguage of $\mathcal{L}_\infty$ that allows $P^{\mathcal{A}}_{\beta}$ only for $\beta < \alpha$, and $P^{\mathcal{A}}_{<\beta}$ only for $\beta \leq \alpha$. Then H_α is just H_∞ restricted to $\mathcal{L}_\alpha$.
The notion $\mathsf{H}_\alpha \overset{\beta}{\vdash} \Gamma$ is used to express that Γ is provable in H_α by a proof of depth less than or equal to β.
We write $\mathsf{H}_\alpha \overset{\beta}{\underset{\star}{\vdash}} \Gamma$ if Γ can be proved by a derivation of depth less than or equal to β only using cuts of the above form.

Lemma

For all Γ:

$$\textit{if} \quad \mathsf{H}_\alpha \overset{\beta}{\vdash} \Gamma, \quad \textit{then} \quad \mathsf{H}_\alpha \overset{<\varepsilon(\beta)}{\underset{\star}{\vdash}} \Gamma.$$

Elimination of one fixed point

Let us study how to eliminate one fixed point via *asymmetric interpretation*.[Can85] Fix an ordinal β.

To see this we introduce a new semiformal system, H'_β, for studying the *stages* of the construction of the fixpoint $P^{\mathcal{A}}_\beta$. The language $\mathcal{L}'_\beta$ is $\mathcal{L}_{<\beta}$ extended by unary predicates $I^{\mathcal{A}}_\gamma$, which we interpret as the stages, so $I^{\mathcal{A}}_0 = \emptyset$ and

$$I^{\mathcal{A}}_\gamma = \{\, x \mid \mathcal{A}(I^{\mathcal{A}}_{<\gamma}, P^{\mathcal{A}}_{<\beta}, x, \beta) \,\}.$$

Elimination of one fixed point, continued

The system extends H'_β extends H_β with the axioms:

$$\Gamma, \neg I_0^{\mathcal{A}}, \qquad \text{for all } \Gamma,$$

and the rules:

Stage rules, successor For all Γ, all closed terms s, and all closed terms t with $|t| = \beta$:

$$\frac{\Gamma, \mathcal{A}(I_\gamma^{\mathcal{A}}, P_{<\beta}^{\mathcal{A}}, s, t)}{\Gamma, I_{\gamma+1}^{\mathcal{A}}(s)} \quad \text{and} \quad \frac{\Gamma, \neg\mathcal{A}(I_\gamma^{\mathcal{A}}, P_{<\beta}^{\mathcal{A}}, s, t)}{\Gamma, \neg I_{\gamma+1}^{\mathcal{A}}(s)}$$

Elimination of one fixed point, continued

Stage rules, limit For all Γ, all closed terms s, and all limit ordinals λ, and all $\gamma < \lambda$:

$$\frac{\Gamma, I_\gamma^{\mathcal{A}}(s)}{\Gamma, I_\lambda^{\mathcal{A}}(s)} \quad \text{and} \quad \frac{\Gamma, \neg I_\delta^{\mathcal{A}}(s) \text{ for all } \delta < \lambda}{\Gamma, \neg I_\lambda^{\mathcal{A}}(s)}$$

Now we introduce the β-rank of formulas A in $\mathcal{L}'_\beta$:

$$\begin{aligned}
\mathrm{rnk}_\beta(A) &= 0, \quad \text{for literals } A \text{ not containing } I_\gamma^{\mathcal{A}}. \\
\mathrm{rnk}_\beta(I_\gamma^{\mathcal{A}}(t)) &= \mathrm{rnk}_\beta(\neg I_\gamma^{\mathcal{A}}(t)) = \omega\gamma \\
\mathrm{rnk}_\beta(B \circ C) &= \max(\mathrm{rnk}_\beta(B), \mathrm{rnk}_\beta(C)) + 1 \\
\mathrm{rnk}_\beta(Qx.\, B) &= \mathrm{rnk}_\beta(B(0)) + 1
\end{aligned}$$

The asymmetric interpretation

Given any ordinals γ, δ with $0 \leq \gamma \leq \delta < \Phi_0$, we define a translation $A^{\gamma,\delta}$ of $\mathcal{L}_\beta$ formulas to $\mathcal{L}'_\beta$:

$$\neg P^{\mathcal{A}}_\beta(s) \mapsto \neg I^{\mathcal{A}}_\gamma(s)$$
$$P^{\mathcal{A}}_\beta(s) \mapsto I^{\mathcal{A}}_\delta(s)$$

Theorem (Asymmetric Interpretation)

Suppose we have a derivation $\mathsf{H}_\beta \overset{\alpha}{\underset{\star}{\vdash}} \Gamma$*. Then we can find uniformly in* $\delta > 0$ *and any sequence* $\boldsymbol{\gamma}$ *below* δ *suitable for* Γ *a derivation:*

$$\mathsf{H}'_\beta \overset{\lambda(\alpha,\boldsymbol{\gamma},\delta)}{\vdash} \Gamma[\boldsymbol{\gamma}, \varphi\alpha(|\boldsymbol{\gamma},\delta| + 1)].$$

Here $\lambda(\alpha, \boldsymbol{\gamma}, \delta) = \omega\varphi\alpha(|\boldsymbol{\gamma}, \delta| + 1) + \omega + 1$*.*

Main Lemma II

Lemma

Assume Γ *is a set of formulas of* $\mathcal{L}_{\beta+\xi}$*. Then we have for all ordinals* ρ *with* $\xi < \omega^{1+\rho}$ *and all ordinals* α*:*

$$\textit{if} \quad \mathsf{H}_{\beta+\omega^{1+\rho}} \overset{\alpha}{\underset{\star}{\vdash}} \Gamma, \quad \textit{then} \quad \mathsf{H}_{\beta+\xi} \overset{\varphi 1 \rho \alpha}{\underset{\star}{\vdash}} \Gamma.$$

Embedding $\widehat{\mathsf{ID}}_\alpha$ in H_α

Lemma

For all $\mathcal{L}_{\mathsf{fix}}$ sentences A we have:

$$\textit{if} \quad \widehat{\mathsf{ID}}_\alpha \vdash A, \quad \textit{then} \quad \mathsf{H}_\alpha \mathrel{\vert\!\!\frac{\;<\varepsilon(\alpha)\;}{\star}} A^\alpha.$$

Main Theorem of JKSS, again

Fix an ordinal $\alpha < \Phi_0$. Let $\varepsilon(\alpha)$ denote the least ε-number greater than α. Define inductively

$$(\alpha \mid 0) := \varepsilon(\alpha), \qquad (\alpha \mid m+1) := \varphi(\alpha \mid m)0.$$

Theorem (Main Theorem)

Let α have Cantor Normal Form

$$\alpha = \omega^{1+\alpha_n} + \cdots + \omega^{1+\alpha_1} + m,$$

where $\alpha_n \geq \cdots \geq \alpha_1$ and $m < \omega$. Then the proof-theoretic ordinal of $\widehat{\mathsf{ID}}_\alpha$ is

$$|\widehat{\mathsf{ID}}_\alpha| = \varphi 1 \alpha_n(\varphi 1 \alpha_{n-1}(\ldots \varphi 1 \alpha_1(\alpha \mid m)) \ldots).$$

Corollaries and connections

Based on the studies of the paper we can add to previous result that $|\widehat{\mathsf{ID}}_{<\omega}| = \Gamma_0$ the following corollary:

Corollary

We have the following proof-theoretical ordinals:

$$
\begin{aligned}
|\widehat{\mathsf{ID}}_{<\omega}| &= \varphi 100 = \Gamma_0, & |\widehat{\mathsf{ID}}_{\omega}| &= \varphi 10\varepsilon_0 = \Gamma_{\varepsilon_0}, \\
|\widehat{\mathsf{ID}}_{<\omega^\omega}| &= \varphi 1\omega 0, & |\widehat{\mathsf{ID}}_{\omega^\omega}| &= \varphi 1\omega\varepsilon_0, \\
|\widehat{\mathsf{ID}}_{<\varepsilon_0}| &= \varphi 1\varepsilon_0 0, & |\widehat{\mathsf{ID}}_{\varepsilon_0}| &= \varphi 1\varepsilon_0\varepsilon_0.
\end{aligned}
$$

Scaling the metapredicative hierarchy

The theories $\widehat{\mathsf{ID}}_\alpha$ can be used to scale (an initial part of) the metapredicative hierarchy. Some results in this direction follow.
From Jäger and Strahm [JS00]:

$$
\begin{aligned}
\widehat{\mathsf{ID}}_w &\equiv \mathsf{ATR}, \\
\widehat{\mathsf{ID}}_{<\omega^\omega} &\equiv \mathsf{ATR}_0 + (\Sigma^1_1\text{-}\mathsf{DC}), \\
\widehat{\mathsf{ID}}_{<\varepsilon_0} &\equiv \mathsf{ATR} + (\Sigma^1_1\text{-}\mathsf{DC}).
\end{aligned}
$$

The autonomous fixed point theories, and further connections with Explicit Mathematics and Martin-Löf type theories to be discussed in later seminars.

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